

# Invisible obstacles <sup>\*†</sup>

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## Abstract

It is proved that one can choose a control function on an arbitrary small open subset of the boundary of an obstacle so that the total radiation from this obstacle for a fixed direction of the incident plane wave and for a fixed wave number will be as small as one wishes. The obstacle is called "invisible" in this case.

## 1 Introduction

Consider a bounded domain  $D \subset \mathbb{R}^n$ ,  $n = 3$ , with a connected Lipschitz boundary  $S$ . Let  $F$  be an arbitrary small, fixed, open subset on  $S$ ,  $F' = S \setminus F$ , and  $N$  be the outer unit normal to  $S$ . The domain  $D$  is the obstacle. Consider the scattering problem:

$$\nabla^2 u + k^2 u = 0 \text{ in } D' := \mathbb{R}^3 \setminus D, \quad u = w \text{ on } F, \quad u_N + hu = 0 \text{ on } F'. \quad (1)$$

Here  $w$  is the function we can set up at will, the control function,  $h$  is a piecewise-continuous function,  $\text{Im } h \geq 0$ , and  $k > 0$  is a fixed constant. The function  $u$  satisfies the following condition:

$$u = u_0 + v, \quad u_0 = e^{ik\alpha \cdot x}, \quad (2)$$

and

$$v = \frac{e^{ikr}}{r} A(\beta, \alpha) + o\left(\frac{1}{r}\right) \quad r := |x| \rightarrow \infty, \quad \beta := \frac{x}{r}. \quad (3)$$

The function  $A(\beta, \alpha)$  is called the scattering amplitude,  $\alpha, \beta \in S^2$  are the unit vectors,  $S^2$  is the unit sphere,  $\alpha$ , the direction of the incident wave  $u_0$ , is assumed fixed, so  $A(\beta, \alpha) = A(\beta)$ . Problem (1)-(3) has a unique solution ([1]).

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<sup>\*</sup>key words: wave scattering, inverse problems, invisible obstacles

<sup>†</sup>AMS subject classification: 35J05, 35R30, 74J20, 74J25; PACS 02.30.Jr, 03.40.K

Define the cross section  $\sigma$ , or the total radiation from the obstacle, as

$$\sigma = \int_{S^2} |A(\beta)|^2 d\beta. \quad (4)$$

The problem is:

*Given an arbitrary small  $\epsilon > 0$ , can one choose  $w$  so that  $\sigma < \epsilon$  ?*

If this choice is possible, we call the obstacle "invisible" for the fixed  $\alpha$  and  $k$ .

Our basic result is the following theorem:

**Theorem 1.** *Given an arbitrary small  $\epsilon > 0$  and an arbitrary small open subset  $F \in S$ , one can find  $w \in C_0^\infty(F)$  such that  $\sigma < \epsilon$ . The same result holds for the boundary conditions  $u|_F = w$ ,  $u|_{F'} = 0$ .*

A similar problem was first posed and solved in [2], where the Neumann boundary condition was assumed and the control function was not  $u$  on  $F$ , but  $u_N$  on  $F$ . The boundary conditions in this paper allow one to consider impedance obstacles, so it broadens the possible applications of our theory. Inverse problems for scattering by obstacles are considered in [1] and [3].

In Section 2 proofs are given.

## 2 Proofs.

### Proof of Theorem 1.

By Green's formula we get

$$v(x) = \int_{F'} G(x, s)(u_{0N} + hu_0)ds + \int_F G_N(x, s)v ds, \quad (5)$$

where  $G$  is the Green's function:

$$\nabla^2 G + k^2 G = -\delta(x - y) \quad \text{in } D', \quad \lim_{|x| \rightarrow \infty} |x| \left( \frac{\partial G}{\partial |x|} - ikG \right) = 0, \quad (6)$$

and

$$G_N + hG = 0 \quad \text{on } F', \quad G = 0 \quad \text{on } F. \quad (7)$$

By Ramm's lemma ([1], p.46), one gets:

$$G(x, y) = \frac{e^{ikr}}{4\pi r} \psi(y, \nu) + o\left(\frac{1}{r}\right), \quad r := |x| \rightarrow \infty, \quad \frac{x}{r} = -\nu. \quad (8)$$

Here  $\psi$  is the scattering solution:

$$\nabla^2 \psi + k^2 \psi = 0 \quad \text{in } D', \quad \psi_N + h\psi = 0 \quad \text{on } F', \quad \psi = 0 \quad \text{on } F, \quad (9)$$

and

$$\psi = e^{ik\nu \cdot x} + \eta, \quad \lim_{|x| \rightarrow \infty} |x|(\eta_r - ik\eta) = 0. \quad (10)$$

Using (4), (5) and (8), we get:

$$A(\beta) = \frac{1}{4\pi} \int_{F'} \psi(s, -\beta)(u_{0N} + hu_0)ds + \frac{1}{4\pi} \int_F (w - u_0)\psi_N(s, -\beta)ds, \quad (11)$$

and

$$\sigma = \int_{S^2} |A_0(\beta) - A_1(\beta)|^2 d\beta, \quad (12)$$

where

$$A_0(\beta) := \frac{1}{4\pi} \int_{F'} \psi(s, -\beta)(u_{0N} + hu_0)ds - \frac{1}{4\pi} \int_F u_0\psi_N(s, -\beta)ds, \quad (13)$$

and

$$A_1(\beta) := \frac{1}{4\pi} \int_F w(s)\psi_N(s, -\beta)ds. \quad (14)$$

The conclusion of Theorem 1 follows immediately from Lemma 1.

**Lemma 1.** *Given an arbitrary function  $f \in L^2(S^2)$  and an arbitrary small  $\epsilon > 0$ , one can find  $w \in C_0^\infty(F)$ , such that  $\|f(\beta) - A_1(\beta)\| < \epsilon$ , where  $\|\cdot\| := \|\cdot\|_{L^2(S^2)}$ .*

Indeed, one can take  $f(\beta) = A_0(\beta)$  and use Lemma 1.

Let us prove Lemma 1.

If this lemma is false, then there is an  $f \in L^2(S^2)$ ,  $f \neq 0$ , such that

$$\int_{S^2} d\beta f(\beta) \int_F ds w(s)\psi_N(s, -\beta) = 0 \quad \forall w \in C_0^\infty(F). \quad (15)$$

This implies

$$\int_{S^2} d\beta f(\beta)\psi_N(s, -\beta) = 0 \quad \forall s \in F. \quad (16)$$

Define the function

$$z(x) := \int_{S^2} d\beta f(\beta)\psi(x, -\beta). \quad (17)$$

This function solves equation

$$\nabla^2 z + k^2 z = 0 \text{ in } D'$$

and satisfies the boundary conditions:

$$z = z_N = 0 \text{ on } F.$$

By the uniqueness of the solution to the Cauchy problem for elliptic equations, this implies

$$z(x) = 0 \quad \text{in } D'. \quad (18)$$

It follows from (18) that  $f = 0$ . This contradiction proves Lemma 1 and, consequently, Theorem 1.

To complete the proof, let us derive from (18) that  $f = 0$ . The function

$$\psi(x, \beta) = T e^{ik\beta \cdot x},$$

where  $T$  is a linear boundedly invertible operator, acting on the  $x$  variable only (see [1]). The specific form of  $T$  is not important for our argument. Applying the inverse operator  $T^{-1}$  to (17) and taking into account (18), one gets:

$$\int_{S^2} d\beta f(\beta) e^{-ik\beta \cdot x} = 0 \quad \forall x \in D'. \quad (19)$$

The left-hand side in (19) is an entire function of  $x$ . Therefore (19) implies

$$\int_{S^2} d\beta f(\beta) e^{-ik\beta \cdot x} = 0 \quad \forall x \in \mathbb{R}^3. \quad (20)$$

Equation (20) means that the Fourier transform of the distribution  $f(\beta) \frac{\delta(|\xi| - k)}{|\xi|^2}$  equals to zero. Here  $\xi = |\xi|\beta$  is the dual to  $x$  Fourier transform variable. By the injectivity of the Fourier transform, it follows that this distribution equals to zero, so  $f = 0$ , and the proof is completed. The last statement of Theorem 1 is proved similarly.  $\square$

### 3 Conclusion

The basic result of this note is the proof of the following statement:

*By choosing a suitable control function on an arbitrarily small open subset of the boundary of a bounded obstacle, one can make the total radiation from this obstacle, although positive, but as small as one wishes, for a fixed wave number and a fixed direction of the incident wave. Thus, the obstacle can be made practically invisible.*

### References

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- [3] Ramm, A.G., **Inverse Problems**, Springer, New York, 2005.